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ROKHLIN ACTIONS OF FINITE GROUPS ON UHF-ABSORBING C^* -ALGEBRAS

SELÇUK BARLAK AND GÁBOR SZABÓ

ABSTRACT. This paper serves as a source of examples of Rokhlin actions or locally representable actions of finite groups on C^* -algebras satisfying a certain UHF-absorption condition. We show that given any finite group G and a separable, unital C^* -algebra A that absorbs $M_{|G|^\infty}$ tensorially, one can lift any group homomorphism $G \rightarrow \text{Aut}(A)/\approx_u$ to an honest Rokhlin action γ of G on A . Unitality may be dropped in favour of stable rank one or being stable. If A belongs to a certain class of C^* -algebras that is classifiable by a suitable invariant (e.g. K -theory), then in fact every G -action on the invariant lifts to a Rokhlin action of G on A . For the crossed product C^* -algebra $A \rtimes_\gamma G$ of a Rokhlin action on a UHF-absorbing C^* -algebra, an inductive limit decomposition is obtained in terms of A and γ . If G is assumed to be abelian, then the dual action $\hat{\gamma}$ is locally representable in a very strong sense. We then show how some well-known constructions of finite group actions with certain predescribed properties can be recovered and extended by the main results of this paper, when paired with known classification theorems. Among these is Blackadar's famous construction of symmetries on the CAR algebra whose fixed point algebras have non-trivial K_1 -groups. Lastly, we use the results of this paper to reduce the UCT problem for separable, nuclear C^* -algebras to a question about certain finite group actions on \mathcal{O}_2 .

0. INTRODUCTION

The study of noncommutative dynamical systems is a very active area of research with lots of open problems and things yet to discover. Be it either the investigation of crossed product C^* -algebras or the actions themselves, C^* -dynamical systems spark great interest from the viewpoint of classification theory. In the particular case of finite group actions, Phillips has been steadily paving the way towards a classification theory of pointwise outer actions on unital Kirchberg algebras, building on ideas of ordinary classification theory of Kirchberg algebras [35, 23] and making use of equivariant absorption theorems in the spirit of [25]. The key ideas of the latter have already been demonstrated in [12]. Concerning the case of poly- \mathbb{Z} groups, initial positive results in this direction are due to Nakamura [30], Izumi-Matui [19] and is currently developed further by Izumi-Matui, see [20].

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An important question related to such a classification theory is how large the range of the objects is that one wishes to classify. For example, if a certain class of C^* -algebras is classified by K -theoretic data and one wishes to study finite group actions on such, this begs the question of whether every group action on the K -theory of a C^* -algebra in this class can be lifted to an honest group action on the C^* -algebra. To be more precise, one poses the following question:

Question. If A belongs to a class of C^* -algebras that is classifiable by a functor Inv (in a suitable sense) and $\sigma : G \curvearrowright \text{Inv}(A)$ is an action of a finite group on the invariant of A , does there exist an action $\alpha : G \curvearrowright A$ with $\text{Inv}(\alpha) = \sigma$?

It seems that, compared to the recent progress in the Elliott classification program, satisfactory answers to this question are very scarce. We note that even within the setting of $G = \mathbb{Z}_n$ and A being an AF algebra, this question appears to be still open, see [4, 10.11.3].

To the authors' best knowledge, only the case of actions on unital UCT Kirchberg algebras has been successfully examined so far, in which the invariant boils down to K -theory. Symmetries, i.e. \mathbb{Z}_2 -actions, were first considered in [2]. The main result asserted that if the C^* -algebras in question are unital UCT Kirchberg algebras in Cuntz standard form, then the above question has an affirmative answer. This was extended to finite cyclic group actions of prime order in [42], where also the assumption of the Cuntz standard form could be dropped. Finally in [22], this was further extended to actions of finite groups whose Sylow subgroups are cyclic.

Certain difficulties in the aforementioned papers arose from the fact that a slightly stronger question was asked than just the above. For example, a related question is if there exists a lift α like in the above question that comes from a (partial) split of the classifying functor applied to σ . Another stronger question, which was answered in [2] for symmetries, is how large the range of actions is with respect to equivariant K -theory instead of ordinary K -theory.

In this paper, we take another viewpoint that is also suitable for actions on not necessarily classifiable C^* -algebras: For a C^* -algebra A and a finite group G , when can a homomorphism $G \rightarrow \text{Aut}(A)/\approx_u$ lift to an honest action of G on A ? While we must certainly impose certain restrictions on A , it turns out that a sufficient criterion is common enough to produce a variety of interesting examples. Incidentally, the lifts we construct all have the Rokhlin property.

Theorem (2.3). *Let G be a finite group and A a separable C^* -algebra that absorbs the UHF algebra $M_{|G|^\infty}$ tensorially. Assume that A is either unital, stable or has stable rank one. Then any homomorphism $G \rightarrow \text{Aut}(A)/\approx_u$ lifts to a Rokhlin action of G on A .*

Notably, no simplicity or classifiability assumption on A is needed in order to prove this. Looking back to the above question about the range of the invariant of G -actions on classifiable C*-algebras, this more general viewpoint is (a priori) weaker because the Elliott invariant alone is usually not strong enough to distinguish between approximate unitary equivalence classes of $*$ -homomorphisms. However, a strong enough classification result paired with UCT usually implies that the canonical map $\text{Aut}(A)/\approx_u \rightarrow \text{Aut}(\text{Inv}(A))$ is not only surjective, but has a split. For example, this works for Kirchberg algebras or simple, nuclear TAF algebras, see [26, 28]. This enables one to reduce the above question about the range of the invariant of actions to the question of being able to lift homomorphisms $G \rightarrow \text{Aut}(A)/\approx_u$ to honest group actions on A . In particular, 2.3 allows us to prove the following result:

Theorem (2.11 and 2.13). *Let G be a finite group and A a separable, unital, nuclear and simple C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A satisfies the UCT and is either purely infinite or TAF. Then Rokhlin actions of G on A exhaust all G -actions on the (ordered) K -theory of A .*

The Rokhlin property for finite groups was pioneered by Herman and Jones in the setting of UHF algebras, see [15, 16]. They followed the argument of Rokhlin actions developed by Connes [6, 7] and Ocneanu [34] for amenable groups in the von Neumann algebra setting. In his remarkable papers [17, 18], Izumi then introduced the Rokhlin property for finite group actions on unital C*-algebras. Finite group actions with the Rokhlin property are very rigid. For example, two Rokhlin actions on a unital, separable C*-algebra, which are pointwise approximately unitarily equivalent, are conjugate by an approximately inner automorphism, see [17]. In fact, the same result holds for non-unital C*-algebras as well; first proved in [31] for C*-algebras of almost stable rank one, and then proved in general in [11]. In particular, this means that up to conjugacy, we must construct every possible Rokhlin action of G on A in order to prove 2.3. This enables us to make some non-trivial observations concerning the structure of the crossed product C*-algebras $A \rtimes_\gamma G$, when γ is an arbitrary action with the Rokhlin property and A absorbs $M_{|G|^\infty}$ tensorially. We will also examine the dual actions of Rokhlin actions more closely, in the case that the acting group is abelian. It turns out that when G and A are as in 2.3 with G abelian and $\gamma : G \curvearrowright A$ has the Rokhlin property, then $\hat{\gamma}$ is locally representable in a very strong sense. In particular, it is an inductive limit type action, where the building blocks and the actions on the building blocks are fairly easy to grasp.

The paper is organized as follows. In the first section, we present some notation and definitions that we will use throughout the paper and remind the reader of some standard techniques.

In the second section, we prove that under the aforementioned conditions on a finite group G and a C*-algebra A , every homomorphism $G \rightarrow$

$\text{Aut}(A)/\approx_u$ lifts to a Rokhlin action of G on A . We do this by constructing an inductive limit model system on which we define the action. Combining UHF-absorption of A with an Elliott intertwining argument, we prove that the model system just recovers the given C^* -algebra A . Several consequences of this model system are deduced, for example an inductive limit decomposition of the crossed product C^* -algebra $A \rtimes_\gamma G$, when A is $M_{|G|^\infty}$ -absorbing and γ is an arbitrary Rokhlin action of G on A .

In the third section, we examine the dual actions of the Rokhlin actions treated in Section 2, when the acting group is abelian. As is known from [17, 31], such a dual action is always approximately representable. Under the UHF-absorption condition, however, it turns out that such a dual action is even locally representable in a very strong sense.

In the fourth section, we treat some interesting examples that arise as consequences of the main results of this paper. It turns out that Blackadar's famous construction of [3] can be recovered and extended to a more general setting by combining 2.13 with Lin's classification theory of TAF algebras [28]. Moreover, an analogous result as the range result for certain symmetries on the CAR algebra [3, 6.2.4] follows for actions of all finite abelian groups, particularly for order p automorphisms for all $p > 2$. In his remarkable work on the Rokhlin property of finite group actions on unital C^* -algebras [17, 18], Izumi gave a range result of approximately representable actions of finite cyclic groups with prime power order on \mathcal{O}_2 , see [17, 4.8(3), 4.9] and [18, 6.4]. This extends quite naturally to finite cyclic groups of any order, and in fact all finite abelian groups, by combining 2.11 with Kirchberg-Phillips classification. On the front of non-unital C^* -algebras, one can also combine 2.3 with Robert's classification theorem [38] to construct actions on stably projectionless C^* -algebras. For example, the construction carried out in [31, 5.6], which was intended as a stably projectionless analogue of [17, 4.7], can be recovered. At last, we sketch how to reduce the UCT problem for separable, nuclear C^* -algebras to the question of whether one can leave the UCT class by passing to crossed product C^* -algebras of \mathcal{O}_2 by finite group actions. We will even reduce it to the more special setting where one only needs to consider certain locally representable actions of \mathbb{Z}_2 and \mathbb{Z}_3 on \mathcal{O}_2 . It is possible that this is already known to some experts (see [4, 23.15.12(d)]), but to the authors' knowledge, there is no proof in the literature.

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1. PRELIMINARIES

Notation 1.1. Unless specified otherwise, we will stick to the following notations throughout the paper.

- G denotes a fixed finite group.
- For a natural number $p \geq 2$, we denote $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.
- A and B are separable C*-algebras.
- \tilde{B} denotes the unitalization of B , if B is non-unital. If B already is unital, then \tilde{B} is B . $\mathcal{M}(B)$ denotes the multiplier algebra of B .
- If M is some set and $F \subset M$ is a finite subset, we write $F \subset\subset M$.
- For $\varepsilon > 0$ and a, b in some normed space, we write $a =_\varepsilon b$ for $\|a - b\| \leq \varepsilon$.

First, we recall some needed definitions. Note that, in the next definition, we are in principle following [24], but with a slightly enhanced notation to point out that we work with ordinary sequence algebras instead of ultrapowers.

Definition 1.2. Let A be a separable C*-algebra. Let $B \subset A_\infty$ be a sub-C*-algebra. Define

$$\text{Ann}(B, A_\infty) = \{x \in A_\infty \mid xb = bx = 0 \text{ for all } b \in B.\}$$

and

$$F_\infty(B, A) = (A_\infty \cap B') / \text{Ann}(B, A_\infty)$$

as the central sequence algebra of A relative to B . One writes $F_\infty(A) = F_\infty(A, A)$. If $\varphi \in \text{Aut}(A)$, we denote the induced automorphism on $F_\infty(A)$, given by componentwise application of φ , by φ_∞ . This notation carries over to (finite) group actions.

Definition 1.3 (following [31, 3.1]). Let G be a finite group and A a separable C*-algebra. Let $\alpha : G \curvearrowright A$ be an action. Then α is said to have the Rokhlin property, if there exists an equivariant and unital *-homomorphism

$$(\mathcal{C}(G), G\text{-shift}) \longrightarrow (F_\infty(A), \alpha_\infty).$$

In this context, we shall also call α a Rokhlin action.

Remark 1.4. Denote by $\{e_g\}_{g \in G}$ the characteristic functions of the points $g \in G$ inside $\mathcal{C}(G)$. By representing the images of each e_g in $F_\infty(A)$ by some bounded sequences in $\ell^\infty(\mathbb{N}, A)$ it is easy to check the following:

Let G be a finite group, A a separable C*-algebra and $\alpha : G \curvearrowright A$ an action via automorphisms. Then α has the Rokhlin property, if and only if the following holds: For all $\varepsilon > 0$ and $F \subset\subset A$, there exist positive contractions $\{a_g\}_{g \in G}$ in A satisfying

- (1) $\left(\sum_{g \in G} a_g\right)x =_\varepsilon x$ for all $x \in F$.
- (2) $\alpha_g(a_h)x =_\varepsilon a_{gh}x$ for all $g, h \in G$ and $x \in F$.
- (3) $\|a_g a_h\| \leq \varepsilon$ for all $g \neq h$ in G and $x \in F$.

$$(4) \quad \|[a_g, x]\| \leq \varepsilon \quad \text{for all } g \in G \text{ and } x \in F.$$

We refer to [41] for a more detailed survey and comparison between several known variants of the Rokhlin property. In particular, it follows from [41] that the Rokhlin elements can actually be chosen in such a way that the relations (2) and (3) hold approximately in norm instead of the strict topology. However, as the above formulation is a priori the weakest version, it is also the easiest to verify.

Definition 1.5. • Two $*$ -homomorphisms $\varphi, \psi : A \rightarrow B$ are called approximately multiplier unitarily equivalent, written $\varphi \approx_{\text{mu}} \psi$, if there are unitaries $u_n \in \mathcal{U}(\mathcal{M}(B))$ such that

$$\psi(a) = \lim_{n \rightarrow \infty} u_n \varphi(a) u_n^* \quad \text{for all } a \in A.$$

- Two $*$ -homomorphisms $\varphi, \psi : A \rightarrow B$ are called approximately unitarily equivalent, written $\varphi \approx_{\text{u}} \psi$, if there are unitaries $u_n \in \mathcal{U}(\tilde{B})$ such that

$$\psi(a) = \lim_{n \rightarrow \infty} u_n \varphi(a) u_n^* \quad \text{for all } a \in A.$$

- A $*$ -endomorphism on A is called approximately inner, if it is approximately unitarily equivalent to id_A . A $*$ -endomorphism on A is called approximately multiplier inner, if it is approximately multiplier unitarily equivalent to id_A .

Remark 1.6. Obviously, the relations $\varphi \approx_{\text{mu}} \psi$ and $\varphi \approx_{\text{u}} \psi$ are the same, if B is unital. In general, the two definitions are different. However, it is known that $\varphi \approx_{\text{mu}} \psi$ implies $\varphi \approx_{\text{u}} \psi$ in the cases that B is either stable or has stable rank one.

It is well-known that $\text{Aut}(A)/\approx_{\text{u}}$ defines a group with the operation $[\theta_1] \cdot [\theta_2] = [\theta_1 \circ \theta_2]$. The same is true for $\text{Aut}(A)/\approx_{\text{mu}}$.

Remark 1.7 (see [43]). (1) Let $q \geq 2$ be any number and $A \cong M_{q^\infty} \otimes A$. Then the canonical embedding

$$\mathbf{1}_{M_{q^\infty}} \otimes \text{id}_A : A \rightarrow M_{q^\infty} \otimes A, \quad a \mapsto \mathbf{1}_{M_{q^\infty}} \otimes a$$

is approximately multiplier unitarily equivalent to an isomorphism.

- (2) Let φ be such an isomorphism. Then one has that

$$\mathbf{1}_{M_{q^n}} \otimes \left(\varphi^{-1} \circ (\mathbf{1}_{M_{q^\infty}} \otimes \text{id}_{M_{q^n} \otimes A}) \right) : M_{q^n} \otimes A \rightarrow M_{q^n} \otimes A$$

is approximately multiplier inner for all $n \in \mathbb{N}$.

Lemma 1.8 (see [39, 2.3.4 and 2.3.3] and [40]). (1) Let A and B be separable C^* -algebras. Let $\varphi_0 : A \rightarrow B$ and $\psi_0 : B \rightarrow A$ be two $*$ -homomorphisms such that $\psi_0 \circ \varphi_0 \approx_{\text{u}} \text{id}_A$ and $\varphi_0 \circ \psi_0 \approx_{\text{u}} \text{id}_B$. Then there is an isomorphism $\varphi : A \rightarrow B$ with $\varphi \approx_{\text{u}} \varphi_0$ and $\varphi^{-1} \approx_{\text{u}} \psi_0$. The same statement holds for \approx_{mu} instead of \approx_{u} .

(2) Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of separable C*-algebras. For every n , let

$$\Phi_n : A_n \rightarrow A_{n+1} \quad \text{and} \quad \Psi_n : A_n \rightarrow A_{n+1}$$

be two *-homomorphisms. If $\Phi_n \approx_u \Psi_n$ for all n , then there exists an isomorphism

$$\Theta : \varinjlim \{A_n, \Psi_n\} \longrightarrow \varinjlim \{A_n, \Phi_n\}$$

such that

$$\Phi_{n+1, \infty} \circ \Psi_n \approx_u \Theta \circ \Psi_{n, \infty} \quad \text{and} \quad \Psi_{n, \infty} \approx_u \Theta^{-1} \circ \Phi_{n, \infty}$$

for all $n \in \mathbb{N}$. The same statement holds for \approx_{mu} instead of \approx_u .

2. AN EXISTENCE THEOREM FOR ROKHLIN ACTIONS ON UHF ABSORBING C*-ALGEBRAS

In this section, we prove that for every finite group G and certain UHF absorbing C*-algebras A , every homomorphism $G \rightarrow \text{Aut}(A)/\approx_u$ lifts to a Rokhlin action of G on A .

Notation 2.1. • For a C*-algebra A and a finite index set I , denote

$$A^{\oplus I} = \bigoplus_{i \in I} A.$$

• For a finite group G , let $\{e_{g,h}\}_{g,h \in G}$ denote the generating matrix units of $M_{|G|}$.

Lemma 2.2. *Let G be a finite group and A a separable C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that either A is unital, stable, or has stable rank one. Let $\{\beta_g\}_{g \in G} \subset \text{Aut}(G)$ be a collection of automorphisms. Let $A^{(n)} = M_{|G|^{n-1}} \otimes A$ and $\beta_g^{(n)} = \text{id}_{M_{|G|^{n-1}}} \otimes \beta_g$ for all $n \in \mathbb{N}, g \in G$ and consider the inductive system*

$$\Psi_n : A^{(n) \oplus G} \rightarrow A^{(n+1) \oplus G}, \quad \Psi_n((x_g)_{g \in G}) = \left(\sum_{h \in G} e_{h,h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)}(x_h) \right)_{g \in G}.$$

Denote $A_\Psi = \varinjlim \{A^{(n) \oplus G}, \Psi_n\}$. Then the embedding

$$\iota_\infty : A \hookrightarrow A_\Psi, \quad x \mapsto \Psi_{1, \infty} \left(((\beta_g)^{-1}(x))_{g \in G} \right)$$

is approximately unitarily equivalent to an isomorphism.

Proof. For every n , define a *-homomorphism

$$P_n : A^{(n) \oplus G} \rightarrow M_{|G|^\infty} \otimes A \quad \text{via} \quad P_n((x_g)_{g \in G}) = \mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{g \in G} e_{g,g} \otimes \beta_g^{(n)}(x_g) \right).$$

Observe for all n and $(x_g)_g \in A^{(n)\oplus G}$ that

$$\begin{aligned}
P_{n+1} \circ \Psi_n((x_g)_{g \in G}) &= P_{n+1} \left(\left(\sum_{h \in G} e_{h,h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)}(x_h) \right)_{g \in G} \right) \\
&= \mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{g \in G} e_{g,g} \otimes \beta_g^{(n+1)} \left(\sum_{h \in G} e_{h,h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)}(x_h) \right) \right) \\
&= \mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{g,h \in G} e_{g,g} \otimes e_{h,h} \otimes \beta_h^{(n)}(x_h) \right) \\
&= P_n((x_g)_{g \in G}).
\end{aligned}$$

Hence, the P_n give rise to a well-defined $*$ -homomorphism $P : A_\Psi \rightarrow M_{|G|^\infty} \otimes A$. By 1.7, we pick an isomorphism $\varphi : A \rightarrow M_{|G|^\infty} \otimes A$ with $\varphi \approx_u \mathbf{1}_{M_{|G|^\infty}} \otimes \text{id}_A$. We obtain a $*$ -homomorphism $p = \varphi^{-1} \circ P : A_\Psi \rightarrow A$.

Now define $\iota : A \hookrightarrow A^{(1)\oplus G}$ via $\iota(x) = ((\beta_g)^{-1}(x))_{g \in G}$. For $g \in G$, let $q_g^{(n)} : A^{(n)\oplus G} \rightarrow A^{(n)}$ denote the canonical projection onto the g -component. Check that

$$\begin{aligned}
p \circ \iota_\infty &= p \circ \Psi_{1,\infty} \circ \iota \\
&= \varphi^{-1} \circ P_1 \circ \iota \\
&= \varphi^{-1} \circ \left(\mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{g \in G} e_{g,g} \otimes \beta_g \circ q_g^{(1)} \circ \iota \right) \right) \\
&= \varphi^{-1} \circ \left(\mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{g \in G} e_{g,g} \otimes \text{id}_A \right) \right) \\
&= \varphi^{-1} \circ (\mathbf{1}_{M_{|G|^\infty}} \otimes \text{id}_A) \\
&\approx_u \text{id}_A.
\end{aligned}$$

Next, we wish to show that also $\iota_\infty \circ p \approx_u \text{id}_{A_\Psi}$. Observe that for all n , we have the identity $\Psi_{1,n} \circ \iota = \mathbf{1}_{M_{|G|^{n-1}}} \otimes \iota$. Using this, we calculate for all n :

$$\begin{aligned}
\Psi_{1,n+1} \circ \iota \circ \varphi^{-1} \circ P_n &= (\mathbf{1}_{M_{|G|^n}} \otimes \iota) \circ \varphi^{-1} \circ P_n \\
&= \bigoplus_{g \in G} (\beta_g^{(n+1)})^{-1} \circ (\mathbf{1}_{M_{|G|^n}} \otimes \text{id}_A) \circ \varphi^{-1} \circ P_n \\
&= \bigoplus_{g \in G} (\beta_g^{(n+1)})^{-1} \circ (\mathbf{1}_{M_{|G|^n}} \otimes \text{id}_A) \circ \varphi^{-1} \circ \left(\mathbf{1}_{M_{|G|^\infty}} \otimes \left(\sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)} \circ q_h^{(n)} \right) \right) \\
&\stackrel{1.7(2)}{\approx_u} \bigoplus_{g \in G} (\beta_g^{(n+1)})^{-1} \circ \left(\sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)} \circ q_h^{(n)} \right) \\
&= \bigoplus_{g \in G} \left(\sum_{h \in G} e_{h,h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)} \circ q_h^{(n)} \right) \\
&= \Psi_n.
\end{aligned}$$

Hence, it is clear that $\iota_\infty \circ p = \Psi_{1,\infty} \circ \iota \circ \varphi^{-1} \circ P \approx_u \text{id}_{A_\Psi}$. The proof is complete with an application of 1.8(1). \square

Using the previous lemma, we can prove an existence result for Rokhlin actions on C*-algebras as in 2.2.

Theorem 2.3. *Let G be a finite group and A a separable C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that either A is unital, stable, or has stable rank one. Let $\{\beta_g\}_{g \in G} \subset \text{Aut}(A)$ be a collection of automorphisms that defines a G -action on A up to approximate unitary equivalence, i.e. $\beta_g \circ \beta_h \approx_u \beta_{gh}$ for all $g, h \in G$. Then there exists a Rokhlin action $\gamma : G \curvearrowright A$ such that $\gamma_g \approx_u \beta_g$ for all $g \in G$.*

Proof. We make use of 2.2 and its proof. Adopt the notation for the $A^{(n)}$, the connecting maps Ψ_n and the limit A_Ψ . Also recall the definition of $\iota : A \rightarrow A^{(1) \oplus G}$ and $\iota_\infty : A \rightarrow A_\Psi$.

To define the action γ , we will use a model system for A similar to A_Ψ . Then we will compare the new model system with A_Ψ and we will see that the action, when pulled back to A , will be an action with the desired properties.

For all n , we define

$$\Phi_n : A^{(n) \oplus G} \rightarrow A^{(n+1) \oplus G} \quad \text{via} \quad \Phi_n((x_g)_{g \in G}) = \left(\sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)}(x_{gh}) \right)_{g \in G}.$$

Consider the limit $A_\Phi = \varinjlim \{A^{(n) \oplus G}, \Phi_n\}$. For all $n \in \mathbb{N}$, we define a G -action $\gamma'^{(n)} : G \curvearrowright A^{(n) \oplus G}$ by the shift $\gamma_f'^{(n)}((x_g)_{g \in G}) = (x_{f^{-1}g})_{g \in G}$. One checks for all n and $f \in G$ that

$$\begin{aligned} \gamma_f'^{(n+1)} \circ \Phi_n((x_g)_{g \in G}) &= \gamma_f'^{(n+1)} \left(\left(\sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)}(x_{gh}) \right)_{g \in G} \right) \\ &= \left(\sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)}(x_{f^{-1}gh}) \right)_{g \in G} \\ &= \Phi_n((x_{f^{-1}g})_{g \in G}) \\ &= \Phi_n \circ \gamma_f'^{(n)}((x_g)_{g \in G}). \end{aligned}$$

Therefore, the actions $\gamma'^{(n)}$ extend to an action $\gamma' : G \curvearrowright A_\Phi$ on the inductive limit. It is obvious that γ' has the Rokhlin property, since the building block actions $\gamma'^{(n)}$ are of the form $\text{id}_{A^{(n)}} \otimes (G\text{-shift}) : G \curvearrowright A^{(n)} \otimes \mathcal{C}(G)$.

Now let $\lambda : G \rightarrow \mathcal{U}(M_{|G|})$ be the left-regular representation of G given by $\lambda(g) = \sum_{h \in G} e_{gh,h}$. Note that $\lambda(g)e_{h,k}\lambda(g)^* = e_{gh,gk}$ for all $g, h, k \in G$.

Then the properties of the β_g show that

$$\begin{aligned}
\Psi_n &\approx_u \text{Ad}\left(\bigoplus_{g \in G} \lambda(g^{-1}) \otimes \mathbf{1}_{A^{(n)}}\right) \circ \Psi_n \\
&= \text{Ad}\left(\bigoplus_{g \in G} \lambda(g^{-1}) \otimes \mathbf{1}_{A^{(n)}}\right) \circ \left(\bigoplus_{g \in G} \sum_{h \in G} e_{h,h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)} \circ q_h^{(n)}\right) \\
&= \bigoplus_{g \in G} \sum_{h \in G} e_{g^{-1}h, g^{-1}h} \otimes (\beta_g^{(n)})^{-1} \circ \beta_h^{(n)} \circ q_h^{(n)} \\
&\approx_u \bigoplus_{g \in G} \sum_{h \in G} e_{g^{-1}h, g^{-1}h} \otimes \beta_{g^{-1}h}^{(n)} \circ q_h^{(n)} \\
&= \bigoplus_{g \in G} \sum_{h \in G} e_{h,h} \otimes \beta_h^{(n)} \circ q_{gh}^{(n)} = \Phi_n
\end{aligned}$$

for all n . So we may apply 1.8(2) to find an isomorphism $\Theta : A_\Psi \rightarrow A_\Phi$ with

$$\Phi_{n+1,\infty} \circ \Psi_n \approx_u \Theta \circ \Psi_{n,\infty} \quad \text{and} \quad \Psi_{n,\infty} \approx_u \Theta^{-1} \circ \Phi_{n,\infty}$$

for all $n \in \mathbb{N}$. Moreover, we apply 2.2 to find an isomorphism $\sigma : A \rightarrow A_\Psi$ with $\sigma \approx_u \iota_\infty$. We define

$$\gamma : G \curvearrowright A \quad \text{by} \quad \gamma_g = \sigma^{-1} \circ \Theta^{-1} \circ \gamma'_g \circ \Theta \circ \sigma \quad \text{for all } g \in G.$$

Since γ' had the Rokhlin property, so has γ . For all $f \in G$, we now observe that

$$\begin{aligned}
\gamma_f &\approx_u \sigma^{-1} \circ \Theta^{-1} \circ \gamma'_f \circ \Theta \circ \Psi_{1,\infty} \circ \iota \\
&\approx_u \sigma^{-1} \circ \Theta^{-1} \circ \gamma'_f \circ \Phi_{2,\infty} \circ \Psi_1 \circ \iota \\
&= \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ \gamma_f^{\prime(2)} \circ \Psi_1 \circ \iota \\
&= \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ \gamma_f^{\prime(2)} \circ (\mathbf{1}_{M_{|G|}} \otimes \iota) \\
&= \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ \gamma_f^{\prime(2)} \circ \left(\bigoplus_{h \in G} \mathbf{1}_{M_{|G|}} \otimes (\beta_h)^{-1}\right) \\
&= \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ \left(\bigoplus_{h \in G} \mathbf{1}_{M_{|G|}} \otimes (\beta_{f^{-1}h})^{-1}\right) \\
&\approx_u \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ \left(\bigoplus_{h \in G} \mathbf{1}_{M_{|G|}} \otimes (\beta_h)^{-1} \circ \beta_f\right) \\
&= \sigma^{-1} \circ \Theta^{-1} \circ \Phi_{2,\infty} \circ (\mathbf{1}_{M_{|G|}} \otimes \iota) \circ \beta_f \\
&\approx_u \sigma^{-1} \circ \Psi_{2,\infty} \circ \Psi_1 \circ \iota \circ \beta_f \\
&= \sigma^{-1} \circ \iota_\infty \circ \beta_f \approx_u \beta_f.
\end{aligned}$$

This finishes the proof. \square

Recall the following particularly useful rigidity result for Rokhlin actions:

Theorem 2.4 (see [17, 3.5], [31, 3.5] and [11, Thm 3.4]). *Let A be a separable C*-algebra. Let $\alpha^{(0)}, \alpha^{(1)} : G \curvearrowright A$ be two Rokhlin actions of a finite group with $\alpha_g^{(0)} \approx_u \alpha_g^{(1)}$ for all $g \in G$. Then there is an approximately inner automorphism $\varphi \in \text{Aut}(A)$ such that $\alpha_g^{(1)} \circ \varphi = \varphi \circ \alpha_g^{(0)}$ for all $g \in G$.*

Definition 2.5. Let H_1, H_2 be two discrete groups. Recall that two group homomorphisms $\psi_1, \psi_2 : H_1 \rightarrow H_2$ are called conjugate, if there exists $g \in H_2$ with $\psi_2(h) = g\psi_1(h)g^{-1}$ for all $h \in H_1$. We denote $\text{Hom}(H_1, H_2)$ modulo the conjugacy relation by $\overline{\text{Hom}}(H_1, H_2)$.

Definition 2.6. Let G be a finite group and A a C*-algebra. Let $\mathcal{R}_G(A)$ denote the set of all Rokhlin actions of G on A and let $\overline{\mathcal{R}}_G(A)$ be the set of all conjugacy classes of Rokhlin actions of G on A .

Corollary 2.7. *Let G be a finite group and A a separable C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is either unital, stable, or has stable rank one. Then the natural map*

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}(G, \text{Aut}(A)/\approx_u), \quad [g \mapsto \alpha_g] \longmapsto [g \mapsto [\alpha_g]_{\approx_u}]$$

is surjective and induces a bijection

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}(G, \text{Aut}(A)/\approx_u).$$

Proof. Surjectivity follows directly from 2.3 and injectivity follows from 2.4. \square

The following fact appears to be well-known among experts, and a special case has been proved in [11]. However, to the best of the authors' knowledge, there is no proof of the following fact in the current literature in this generality, so we shall give one here for the reader's convenience. Note that this rounds out the above result 2.7 quite nicely.

Proposition 2.8. *Let G be a finite group and A a separable C*-algebra. If there exists a Rokhlin action $\alpha : G \curvearrowright A$ such that α_g is approximately inner for all g , then $A \cong M_{|G|^\infty} \otimes A$.*

Proof. Let A^1 and $(A_\infty)^1$ denote the unitalization of A and A_∞ , respectively. Since α_g is approximately inner for every $g \in G$, we find a family of unitaries $\{u_g\}_{g \in G} \subset (A_\infty)^1$ satisfying

$$\alpha_g(a) = u_g a u_g^* \quad \text{for all } a \in A.$$

Let $\varepsilon : (A_\infty)^1 \rightarrow \mathbb{C}$ denote the canonical character, and define

$$C := C^*(A, \{u_g - \varepsilon(u_g) \mid g \in G\}) \subset A_\infty.$$

As C is separable and α has the Rokhlin property, one can apply 1.4 to construct a family of positive contractions

$$\{f_g\}_{g \in G} \subset A_\infty \cap C' = A_\infty \cap (A \cup \{u_g \mid g \in G\})'$$

such that for all $g, h, k \in G$ with $h \neq k$ and $c \in C$, the following relations hold:

$$\left(\sum_{g \in G} f_g\right)c = c, \quad \alpha_{g,\infty}(f_h)c = f_{gh}c, \quad f_h f_k c = 0, \quad \text{and} \quad [c, f_g] = 0.$$

Using again that α is pointwise approximately inner, we find a family of unitaries $\{v_g\}_{g \in G} \subset (A_\infty)^1$ with the property that

$$\alpha_{g,\infty}(a) = v_g a v_g^* \quad \text{for all } a \in A \cup \{f_g \mid g \in G\}.$$

For $g \in G$, we define

$$d_g := u_g^* v_g f_1^{1/2} \in A_\infty \cap A'.$$

Note that d_g indeed commutes with all elements of A , since this holds for both f_1 and $u_g^* v_g$. The latter is true as both u_g and v_g implement the automorphism α_g on $A \subset A_\infty$. Moreover, for every $a \in A$ and $g \in G$, one computes

$$d_g d_g^* a = u_g^* v_g f_1 v_g^* u_g a = u_g^* f_g u_g a = f_g a,$$

and

$$d_g^* d_g a = f_1^{1/2} v_g^* u_g u_g^* v_g f_1^{1/2} a = f_1 a.$$

For every $g \in G$, we obtain well-defined elements

$$x_g := d_g + \text{Ann}(A, A_\infty), \quad e_g := f_g + \text{Ann}(A, A_\infty) \in F_\infty(A).$$

The family $\{e_g\}_{g \in G}$ then defines a partition of unity consisting of projections in $F_\infty(A)$. Furthermore, for every $g \in G$, we have $x_g x_g^* = e_g$ and $x_g^* x_g = e_1$ by construction. Using that the universal unital C^* -algebra

$$D := C^*(y_g \text{ partial isometry} \mid y_g^* y_g = y_1^* y_1, \sum_{g \in G} y_g y_g^* = 1)$$

is isomorphic to $M_{|G|}$ via

$$D \longrightarrow M_{|G|}, \quad y_g \mapsto e_{g,1},$$

one concludes that the elements $\{x_g\}_{g \in G}$ give rise to a unital $*$ -homomorphism

$$M_{|G|} \longrightarrow F_\infty(A).$$

A standard reindexation argument shows that then there exists also a unital $*$ -homomorphism

$$M_{|G|^\infty} \longrightarrow F_\infty(A).$$

By applying either [24, 4.11] or [43, 2.3], we get $A \cong A \otimes M_{|G|^\infty}$. \square

Remark 2.9. We have proved the surjectivity part of 2.7, namely 2.3, by combining $M_{|G|^\infty}$ -absorption of the given C*-algebra A with an Elliott intertwining argument (see 1.8), using approximate unitary equivalence in the sense of 1.5 with unitaries in \tilde{A} . One can also apply Elliott intertwining with multiplier approximate unitary equivalence in the same manner, to carry out the constructions of 2.2 and 2.3 in the case of arbitrary separable C*-algebras. This way, one can prove the following weakening of 2.7 for every separable C*-algebra A :

If $A \cong M_{|G|^\infty} \otimes A$, then the natural map

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}(G, \text{Aut}(A)/\approx_{\text{mu}}), \quad [g \mapsto \alpha_g] \longmapsto [g \mapsto [\alpha_g]_{\approx_{\text{mu}}}]$$

is surjective.

However, it remains unclear if Rokhlin actions in the sense of 1.3 can exhibit rigidity like in 2.4 with respect to multiplier approximate unitary equivalence. Therefore, it is also unclear whether one can always expect a naturally induced bijection between $\overline{\mathcal{R}}_G(A)$ and the corresponding $\overline{\text{Hom}}$ -set, as in 2.7.

Remark. In the unital case of 2.7, it has been shown in [37, 5.17] that actions with the Rokhlin property are generic. However, such density results are usually not very helpful for finding examples of actions satisfying some prescribed properties, since it is not even clear how large the set of ordinary actions is. For actions on UHF-absorbing classifiable C*-algebras, the following consequences of 2.7 should be satisfactory results in that direction:

Corollary 2.10. *Let G be a finite group and A a C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is expressible as an inductive limit of 1-NCCW complexes that have trivial K_1 -groups. For convenience, let $\text{Aut}_+(\text{Cu}^\sim(A))$ denote the set of those automorphisms $\text{Cu}^\sim(A) \rightarrow \text{Cu}^\sim(A)$ that send the class of a strictly positive element in A to a class of a strictly positive element. Then the natural map*

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}\left(G, \text{Aut}_+(\text{Cu}^\sim(A))\right), \quad [g \mapsto \alpha_g] \longmapsto [g \mapsto \text{Cu}^\sim(\alpha_g)]$$

is surjective and induces a bijection

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}\left(G, \text{Aut}(\text{Cu}^\sim(A))\right).$$

Proof. This follows directly from 2.7 combined with Robert's classification theorem [38, 1.0.1]. \square

Corollary 2.11. *Let G be a finite group and A a unital UCT Kirchberg algebra with $A \cong M_{|G|^\infty} \otimes A$. Then the natural map*

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}\left(G, \text{Aut}((K_0(A), [\mathbf{1}_A], K_1(A)))\right), \quad [g \mapsto \alpha_g] \longmapsto [g \mapsto K_*(\alpha_g)]$$

is surjective and induces a bijection

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}\left(G, \text{Aut}\left((K_0(A), [\mathbf{1}_A], K_1(A))\right)\right).$$

Proof. It is well-known that $*$ -homomorphisms between Kirchberg algebras are classified via KL -theory, see [39, Section 2.5]. In particular, it is known that the natural map

$$\text{Aut}(A)/\approx_u \rightarrow \{\kappa \in KL(A, A)^{-1} \mid \kappa_0([\mathbf{1}_A]_0) = [\mathbf{1}_A]_0\}$$

is a group isomorphism. On the other hand, the UCT assumption means that the natural map

$$KL(A, A)^{-1} \rightarrow \text{Aut}((K_0(A), K_1(A)))$$

is a split-surjective group homomorphism, see [4, 23.11.1]. (Keep in mind that KL is defined as a quotient of KK .) Therefore, the natural map

$$\text{Aut}(A)/\approx_u \rightarrow \text{Aut}((K_0(A), [\mathbf{1}_A], K_1(A)))$$

is also split-surjective. In particular, group actions of G on the Elliott invariant always lift to homomorphisms $G \rightarrow \text{Aut}(A)/\approx_u$. Hence the surjectivity of the map

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}\left(G, \text{Aut}((K_0(A), [\mathbf{1}_A], K_1(A)))\right)$$

follows directly from 2.7. The injectivity of the map

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}\left(G, \text{Aut}((K_0(A), [\mathbf{1}_A], K_1(A)))\right)$$

follows from [18, 4.2]. \square

Remark 2.12. It is possible to prove a result similar to 2.11 that does not require the assumption of the UCT. The following holds:

Let G be a finite group and A a unital Kirchberg algebra with $A \cong M_{|G|^\infty} \otimes A$. Let $\kappa : G \rightarrow \{x \in KK(A, A)^{-1} \mid x_0([\mathbf{1}]_0) = [\mathbf{1}]_0\}$ be a group homomorphism. Then there exists a Rokhlin action $\gamma : G \curvearrowright A$ with $KK(\gamma) = \kappa$.

Proof. A rough sketch of the proof goes as follows. Using Kirchberg-Phillips classification, one lifts κ to a family $\{\beta_g\}_{g \in G} \subset \text{Aut}(A)$ that defines a G -action on the level of KK . With this family of automorphisms, define the inductive system $\{A^{(n) \oplus G}, \Phi_n\}$ just as in the proof of 2.3. Making use of the unique $|G|$ -divisibility of $KK(A, B)$ for any separable C^* -algebra B and plugging in the Milnor sequence (see [4, 21.5.2]) for the functor $KK(-, B)$, one can calculate that the map

$$j : A \hookrightarrow A_\Phi, \quad x \mapsto \Phi_{1, \infty}\left(\left((\beta_g)^{-1}(x)\right)_{g \in G}\right)$$

yields a KK -equivalence, and is hence asymptotically unitarily equivalent to an isomorphism σ . One defines the action $\gamma' : G \curvearrowright A_\Phi$ just as in the proof of 2.3 and pulls it back via σ , i.e. $\gamma = \sigma^{-1} \circ \gamma' \circ \sigma$. Finally, one calculates that

$KK(\gamma'_g \circ j) = KK(j \circ \beta_g)$ for all $g \in G$, and hence $KK(\gamma_g) = KK(\beta_g) = \kappa(g)$ for all $g \in G$. \square

Corollary 2.13. *Let G be a finite group and A a separable, unital, simple and nuclear C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is TAF and satisfies the UCT. Then the natural map*

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}\left(G, \text{Aut}\left((K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A))\right)\right)$$

is surjective and induces a bijection

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}\left(G, \text{Aut}\left((K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A))\right)\right).$$

Proof. It is known from Lin's classification of TAF algebras [28] that *-homomorphisms between separable, unital, simple and nuclear TAF algebras are classified via KL -theory. In particular, the natural map

$$\text{Aut}(A)/\approx_u \rightarrow \{\kappa \in KL(A, A)^{-1} \mid \kappa_0([\mathbf{1}_A]_0) = [\mathbf{1}_A]_0, \kappa_0(K_0(A)^+) = K_0(A)^+\}$$

is a group isomorphism. On the other hand, the UCT assumption means that the natural map

$$KL(A, A)^{-1} \rightarrow \text{Aut}((K_0(A), K_1(A)))$$

is a split-surjective group homomorphism, see [4, 23.11.1]. (Keep in mind that KL is defined as a quotient of KK .) Therefore, the natural map

$$\text{Aut}(A)/\approx_u \rightarrow \text{Aut}\left((K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A))\right)$$

is also split-surjective. In particular, group actions of G on the Elliott invariant always lift to homomorphisms $G \rightarrow \text{Aut}(A)/\approx_u$. Hence the surjectivity of the map

$$\mathcal{R}_G(A) \longrightarrow \text{Hom}\left(G, \text{Aut}\left((K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A))\right)\right)$$

follows directly from 2.7. Injectivity of

$$\overline{\mathcal{R}}_G(A) \longrightarrow \overline{\text{Hom}}\left(G, \text{Aut}\left((K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A))\right)\right)$$

follows from [18, 4.3]. \square

Remark 2.14. We conjecture that the statement analogous to 2.13 is true for TAI algebras, TASI algebras and most generally the recently classified C*-algebras of generalized tracial rank at most one, which exhaust the entire Elliott invariant for weakly unperforated K_0 , see [13].

Let Ell denote the full Elliott functor. At least the surjectivity part boils down to the question if in the UCT case, the natural map

$$\text{Aut}(A)/\approx_u \longrightarrow \text{Aut}(\text{Ell}(A))$$

is split-surjective. In the TAF case, this is fairly clear because the Elliott invariant only involves (ordered) K -theory. The UCT then implies the existence of a splitting map from the Hom-sets of K -theory back to KK -theory.

For TAI algebras, this map seems to be split-surjective as well, see [29]. In personal communication, Niu has confirmed to us that this map is in fact split-surjective for all separable, unital, nuclear, simple C^* -algebras with generalized tracial rank at most one and satisfying the UCT.

Next, we show how 2.3 helps to obtain a general inductive limit decomposition for crossed products of Rokhlin actions on UHF-absorbing C^* -algebras.

Theorem 2.15. *Let G be a finite group and A a separable C^* -algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is either unital, stable or has stable rank one. Let $\alpha : G \curvearrowright A$ be a Rokhlin action. Let $A^{(n)} = M_{|G|^{n-1}} \otimes A$ and $\alpha_g^{(n)} = \text{id}_{M_{|G|^{n-1}}} \otimes \alpha_g$ for all $n \in \mathbb{N}, g \in G$ and consider the inductive system*

$$\Theta_n : A^{(n+1)} \rightarrow A^{(n+2)}$$

defined by

$$\Theta_n \left(\sum_{g,f \in G} e_{g,f} \otimes x_{g,f} \right) = \sum_{g,f \in G} e_{g,f} \otimes \sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n)}(x_{gh,fh}).$$

Then $A_\Theta := \varinjlim \{A^{(n+1)}, \Theta_n\}$ is isomorphic to $A \rtimes_\alpha G$. In particular, $A \rtimes_\alpha G$ can be expressed as an inductive limit whose building blocks are isomorphic to A .

Proof. Recall the notation from 2.3 and its proof. For $\beta_g = \alpha_g$, define the connecting maps

$$\Phi_n : A^{(n) \oplus G} \rightarrow A^{(n+1) \oplus G}, \quad \Phi_n((x_g)_{g \in G}) = \left(\sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n)}(x_{gh}) \right)_{g \in G}$$

and consider $A_\Phi = \varinjlim \{A^{(n) \oplus G}, \Phi_n\}$. Combining the proof of 2.3 with 2.4 yields that (A, α) is conjugate to (A_Φ, γ) , where $\gamma : G \curvearrowright A_\Phi$ is given on each building block by

$$\gamma^{(n)} : G \curvearrowright A^{(n) \oplus G}, \quad \gamma_f^{(n)}((x_g)_{g \in G}) = (x_{f^{-1}g})_{g \in G}.$$

Hence it suffices to show that $A_\Phi \rtimes_\gamma G \cong A_\Theta$.

Let $\lambda : G \rightarrow \mathcal{U}(M_{|G|})$ be the left regular representation, i.e. $\lambda(g) = \sum_{h \in G} e_{gh,h}$ for all $g \in G$. For each n , define

$$\varphi^{(n)} : A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G \rightarrow A^{(n+1)}, \quad \sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} \text{diag}(a_g) \cdot \lambda(g).$$

It is obvious that $\varphi^{(n)}$ is an isomorphism, since it is just a variant of the natural covariant representation coming from the left regular representation.

Moreover, one can check that the following diagram commutes for all n :

$$\begin{array}{ccc} A^{(n)\oplus G} \rtimes_{\gamma^{(n)}} G & \xrightarrow{\Phi_n \rtimes G} & A^{(n+1)\oplus G} \rtimes_{\gamma^{(n+1)}} G \\ \varphi^{(n)} \downarrow & & \downarrow \varphi^{(n+1)} \\ A^{(n+1)} & \xrightarrow{\Theta_n} & A^{(n+2)} \end{array}$$

Hence one gets

$$A_\Phi \rtimes_\gamma G \cong \varinjlim \left\{ A^{(n)\oplus G} \rtimes_{\gamma^{(n)}} G, \Phi_n \rtimes G \right\} \cong \varinjlim \left\{ A^{(n+1)}, \Theta_n \right\} = A_\Theta.$$

□

With the help of the above model system for Rokhlin actions on UHF-absorbing C*-algebras, one also observes quite easily that the crossed product is always isomorphic to the fixed point algebra.

Theorem 2.16. *Let G be a finite group and A a separable C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is either unital, stable or has stable rank one. Let $\alpha : G \curvearrowright A$ be a Rokhlin action. Then $A^\alpha \cong A \rtimes_\alpha G$.*

Proof. Recall the notation for $A^{(n)}, \alpha_g^{(n)}, \gamma^{(n)}$ and the maps $\Phi_n : A^{(n)\oplus G} \rightarrow A^{(n+1)\oplus G}$, $\Theta_n : A^{(n+1)} \rightarrow A^{(n+2)}$ from 2.3 and 2.15. We have seen that

$$(A, \alpha) = (A_\Phi, \gamma) = \varinjlim \left\{ (A^{(n)\oplus G}, \gamma^{(n)}), \Phi_n \right\}$$

and

$$A \rtimes_\alpha G \cong A_\Theta = \varinjlim \left\{ A^{(n+1)}, \Theta_n \right\}.$$

From the first (equivariant) inductive limit decomposition, it follows that

$$A^\alpha \cong \varinjlim \left\{ (A^{(n)\oplus G})^{\gamma^{(n)}}, \Phi_n \right\}.$$

The building blocks are obviously just isomorphic copies of $A^{(n)}$ as every element $(x_g)_g \in A^{(n)\oplus G}$, which is fixed by the shift action $\gamma^{(n)}$, has the same entry everywhere. The connecting map Φ_n then restricts to

$$\Omega_n : A^{(n)} \rightarrow A^{(n+1)}, \quad \Omega_n(x) = \sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n)}(x).$$

Let σ be the flip automorphism on $M_{|G|} \otimes M_{|G|}$. Recall also the right-regular representation $\rho : G \rightarrow \mathcal{U}(M_{|G|})$ given by $\rho(g) = \sum_{h \in G} e_{hg^{-1}, h}$. Then we see

for every $n \in \mathbb{N}$ and $x = (x_{g,f})_{g,h \in G} \in A^{(n+1)}$ that

$$\begin{aligned}
& \Theta_n(x) \\
&= \sum_{g,f \in G} e_{g,f} \otimes \sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n)}(x_{gh,fh}) \\
&= (\sigma \otimes \text{id}_{A^{(n)}}) \left(\sum_{h \in G} e_{h,h} \otimes \sum_{g,f \in G} e_{g,f} \otimes \alpha_h^{(n)}(x_{gh,fh}) \right) \\
&= (\sigma \otimes \text{id}_{A^{(n)}}) \left(\sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n+1)} \left(\sum_{g,f \in G} e_{gh^{-1},fh^{-1}} \otimes x_{g,f} \right) \right) \\
&= (\sigma \otimes \text{id}_{A^{(n)}}) \left(\sum_{h \in G} e_{h,h} \otimes \alpha_h^{(n+1)} \circ \text{Ad}(\rho(h) \otimes \mathbf{1}_{\mathcal{M}(A^{(n)})}) \left(\sum_{g,f \in G} e_{g,f} \otimes x_{g,f} \right) \right) \\
&= (\sigma \otimes \text{id}_{A^{(n)}}) \circ \text{Ad} \left(\sum_{h \in G} e_{h,h} \otimes \rho(h) \otimes \mathbf{1}_{\mathcal{M}(A^{(n)})} \right) \circ \Omega_{n+1}(x).
\end{aligned}$$

Now the flip automorphism σ is well-known to be inner. The unitary implementing it is

$$\sum_{g,h \in G} e_{g,h} \otimes e_{h,g} \in M_{|G|} \otimes M_{|G|}.$$

In particular, Θ_n is unitarily equivalent to Ω_{n+1} for all n . By 1.8(2), this yields an isomorphism between the inductive limits, so in particular between A^α and $A \rtimes_\alpha G$. \square

3. LOCAL REPRESENTABILITY OF DUALS OF ROKHLIN ACTIONS ON UHF-ABSORBING C^* -ALGEBRAS

In this section, we look at actions of finite abelian groups. Particularly, we take a look at the class of Rokhlin actions treated in the previous section, and will describe the dual actions in terms of local representability.

Remark 3.1. If $B \subset A$ is a non-degenerate sub- C^* -algebra, then one can see easily for every $b, c \in A_\infty \cap B'$ that $b + \text{Ann}(B, A_\infty) = c + \text{Ann}(B, A_\infty)$ implies $ba = ca$ and $ab = ac$ for all $a \in A$. Thus it makes sense to multiply elements of $F_\infty(B, A)$ with elements of A to obtain elements in A_∞ . For example,

$$A \longrightarrow A_\infty, \quad a \mapsto cac^*$$

yields a well-defined c.p.c. map for all contractions $c \in F_\infty(B, A)$. If c happens to be a unitary in $F_\infty(B, A)$, then this defines a $*$ -homomorphism, whose restriction to B is just the canonical inclusion $B \subset A \hookrightarrow A_\infty$.

Definition 3.2 (cf. [17, Section 3.2] and [31, Section 4]). Let A be a separable C*-algebra, G a finite abelian group and $\alpha : G \curvearrowright A$ an action.

- (1) α is called approximately representable, if there is a unitary representation $w : G \rightarrow F_\infty(A^\alpha, A^\alpha) \subset F_\infty(A^\alpha, A)$ with $\alpha_g(a) = w(g)aw(g)^*$ for all $a \in A$ and $g \in G$.
- (2) α is called locally representable, if there exists a sequence of C*-subalgebras $A_n \subset A$ with $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_n A_n}$ and unitary representations $w_n : G \rightarrow \mathcal{U}(\mathcal{M}(A_n))$ such that

$$\alpha_g|_{A_n} = \text{Ad}(w_n(g)) \quad \text{for all } g \in G \text{ and } n \in \mathbb{N}.$$

- (3) Let \mathfrak{C} be a class of separable C*-algebras. α is called locally \mathfrak{C} -representable, if it is locally representable and the A_n from (2) may be chosen to be isomorphic to a C*-algebra in \mathfrak{C} .

Recall the following duality between Rokhlin actions and approximately representable actions:

Theorem 3.3 (see [17, 3.8] and [31, 4.4]). *Let α be an action of a finite abelian group G on a separable C*-algebra A . Then*

- (i) α has the Rokhlin property iff $\hat{\alpha}$ is approximately representable.
- (ii) α is approximately representable iff $\hat{\alpha}$ has the Rokhlin property.

For Rokhlin actions on UHF-absorbing C*-algebras, we can partially strengthen the assertion about its dual action:

Theorem 3.4. *Let G be a finite abelian group and A a separable C*-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is either unital, stable, or has stable rank one. Let $\alpha : G \curvearrowright A$ be a Rokhlin action. Then $\hat{\alpha} : \hat{G} \curvearrowright A \rtimes_\gamma G$ is locally $\{A\}$ -representable.*

Proof. Recall the notation from 2.3 and 2.15. In particular, the definition of the connecting maps

$$\Phi_n : A^{(n) \oplus G} \rightarrow A^{(n+1) \oplus G} \quad \text{and} \quad \Theta_n : A^{(n+1)} \rightarrow A^{(n+2)}.$$

Combining the proof of 2.3 with 2.4 yields that (A, α) is conjugate to (A_Φ, γ) , where $\gamma : G \curvearrowright A_\Phi$ is given on each building block by

$$\gamma^{(n)} : G \curvearrowright A^{(n) \oplus G}, \quad \gamma_f^{(n)}((x_g)_{g \in G}) = (x_{f^{-1}g})_{g \in G}.$$

Moreover, recall the isomorphisms

$$\varphi^{(n)} : A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G \rightarrow A^{(n+1)}, \quad \sum_{g \in G} a_g u_g \mapsto \sum_{g \in G} \text{diag}(a_g) \cdot S(g).$$

that made the following diagram commutative for all n :

$$\begin{array}{ccc} A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G & \xrightarrow{\Phi_n \rtimes G} & A^{(n+1) \oplus G} \rtimes_{\gamma^{(n+1)}} G \\ \varphi^{(n)} \downarrow & & \downarrow \varphi^{(n+1)} \\ A^{(n+1)} & \xrightarrow{\Theta_n} & A^{(n+2)} \end{array}$$

For each n , we define the action $\delta^{(n)} : \widehat{G} \curvearrowright A^{(n+1)}$ induced by the unitary representation

$$\widehat{G} \rightarrow \mathcal{U}(\mathcal{M}(A^{(n+1)})), \quad \chi \mapsto U(\chi) \otimes \mathbf{1}_{\mathcal{M}(A^{(n)})} \quad \text{with} \quad U(\chi) = \sum_{h \in G} \chi(h) \cdot e_{h,h}.$$

As one easily checks that

$$U(\chi) \lambda(g) U(\chi)^* = \chi(g) \cdot \lambda(g) \quad \text{for all } g \in G, \chi \in \widehat{G},$$

one gets that the diagram

$$\begin{array}{ccc} A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G & \xrightarrow{\widehat{\gamma^{(n)}} \chi} & A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G \\ \varphi^{(n)} \downarrow & & \downarrow \varphi^{(n)} \\ A^{(n+1)} & \xrightarrow{\delta_\chi^{(n)}} & A^{(n+1)} \end{array}$$

commutes for all n and $\chi \in \widehat{G}$. In particular, the building block actions $\delta^{(n)}$ give rise to an action $\delta : \widehat{G} \curvearrowright A_\Theta$.

The two commuting diagrams above yield equivariant inductive limit decompositions

$$\begin{aligned} (A_\Phi \rtimes_\gamma G, \hat{\gamma}) &\cong \varinjlim \left\{ (A^{(n) \oplus G} \rtimes_{\gamma^{(n)}} G, \widehat{\gamma^{(n)}}), \Phi_n \rtimes G \right\} \\ &\cong \varinjlim \left\{ (A^{(n+1)}, \delta^{(n)}), \Theta_n \right\} \\ &\cong (A_\Theta, \delta). \end{aligned}$$

Since $(A, \alpha) \cong (A_\Phi, \gamma)$, the action $\hat{\alpha}$ is conjugate to δ . As δ is clearly locally $\{A\}$ -representable, this finishes the proof. \square

4. EXAMPLES AND APPLICATIONS

In this section, we give some applications to the results that have been proved in the previous sections.

Recall the following useful result due to Izumi, which simplifies the computation of the K -theory of crossed products by finite group actions with the Rokhlin property:

Theorem 4.1 (see [17, 3.13]). *Let A be a simple unital C^* -algebra, G a finite group and $\alpha : G \curvearrowright A$ a Rokhlin action. Then*

$$K_i(A^\alpha) = \bigcap_{g \in G} \ker(\text{id} - K_i(\alpha_g)) \quad \text{for } i = 0, 1$$

and the inclusion $A^\alpha \hookrightarrow A$, on the level of K -theory, coincides with the canonical inclusion of this subgroup.

The following recovers and generalizes [3, 6.3.4] by combining 2.13 with known classification results.

Example 4.2. Let (G_0, G_0^+, u) be a countable, uniquely 2-divisible scaled ordered abelian group, which is weakly unperforated and has the Riesz interpolation property. Let σ be a scaled ordered group automorphism of order 2 on G_0 , such that $\ker(\text{id} - \sigma)$ is isomorphic to $(\mathbb{Z}[\frac{1}{2}], \mathbb{Z}[\frac{1}{2}]^+, 1)$ as an ordered group with order unit. Let G_1 be any countable, uniquely 2-divisible abelian group. Then there exists a \mathbb{Z}_2 -action γ on M_{2^∞} such that $M_{2^\infty}^\gamma$ is a simple UCT TAF C*-algebra with $K_0(M_{2^\infty}^\gamma) \cong G_0$ as ordered groups and $K_1(M_{2^\infty}^\gamma) \cong G_1$. Moreover, γ is locally $\{M_{2^\infty}^\gamma\}$ -representable.

Proof. By combining the range result [10] with [27], there exists a separable, unital, nuclear, simple C*-algebra A with tracial rank zero and satisfying the UCT such that

$$(K_0(A), K_0(A)^+, [\mathbf{1}_A], K_1(A)) = (G_0, G_0^+, u, G_1).$$

We make use of the classification theorem of [28]. As the K -groups of A are uniquely 2-divisible, we have $A \cong M_{2^\infty} \otimes A$.

Consider the \mathbb{Z}_2 -action on the above quadruple of A whose K_0 -part agrees with σ , and the K_1 -part acts as multiplication by -1 . By 2.13, we can lift this to a Rokhlin action $\alpha : \mathbb{Z}_2 \curvearrowright A$ with $K_0(\alpha) = \sigma$ and $K_1(\alpha) = -\text{id}$. Applying 4.1 yields that the Elliott invariant of $A \rtimes_\alpha \mathbb{Z}_2$ is the same as that of the CAR algebra M_{2^∞} . As taking crossed products with Rokhlin actions preserves the property of being TAF (see [36, 2.6]) and satisfying the UCT (see [41, Section 5]), $A \rtimes_\alpha \mathbb{Z}_2$ is again TAF and satisfies the UCT, and hence $A \rtimes_\alpha \mathbb{Z}_2 \cong M_{2^\infty}$ by [28]. Under this identification, the dual action $\hat{\alpha}$ gives rise to a locally $\{A\}$ -representable (see 3.4) action $\gamma : \mathbb{Z}_2 \curvearrowright M_{2^\infty}$ with

$$M_{2^\infty}^\gamma \cong (A \rtimes_\alpha \mathbb{Z}_2)^{\hat{\alpha}} \cong A.$$

□

Remark. An analogous statement is true for actions of any finite abelian group. We state it in the next example for \mathbb{Z}_p instead of \mathbb{Z}_2 . As the proof is very similar to the above, we omit it.

Example 4.3. Let $p \geq 2$ be a natural number. Let (G_0, G_0^+, u) be a countable, uniquely p -divisible ordered abelian group with order unit, which is weakly unperforated and has the Riesz interpolation property. Let σ_0 be an ordered group automorphism of order p on G_0 , such that $\ker(\text{id} - \sigma_0)$ is isomorphic to $(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}[\frac{1}{p}]^+, 1)$ as an ordered group with order unit. Let G_1 be a countable, uniquely p -divisible abelian group with an order p automorphism over σ_1 such that $\ker(\text{id} - \sigma_1) = 0$. Then there exists a \mathbb{Z}_p -action γ on M_{p^∞} such that $M_{p^\infty}^\gamma$ is a simple UCT TAF C*-algebra with $K_0(M_{p^\infty}^\gamma) \cong G_0$ as ordered groups and $K_1(M_{p^\infty}^\gamma) \cong G_1$. Moreover, γ can be chosen to be locally $\{M_{p^\infty}^\gamma\}$ -representable.

Next, we shift our focus in the direction of UHF-absorbing (UCT) Kirchberg algebras. As it turns out, the range results of [17, 4.8(3), 4.9] and [18, 6.4] about approximately representable actions of cyclic groups of prime

power cardinality on \mathcal{O}_2 can be recovered and extended by combining 2.11 with Kirchberg-Phillips classification. (We note, however, that in the following more general cases, this does not necessarily cover the whole range.) Moreover, just as for 4.3, an analogous result holds for all finite abelian groups, but we only state it for the cyclic case:

Example 4.4. Let $p \geq 2$ be a natural number. For $i = 0, 1$, let G_i be a countable, uniquely p -divisible abelian group with an order p automorphism $\sigma_i \in \text{Aut}(G_i)$ satisfying $\ker(\text{id} - \sigma_i) = 0$. Then there exists an action $\gamma : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ such that \mathcal{O}_2^γ is a unital UCT Kirchberg algebra with

$$K_i(\mathcal{O}_2^\gamma) \cong K_i(\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_p) \cong G_i.$$

Moreover, γ is locally $\{\mathcal{O}_2^\gamma\}$ -representable. The automorphisms σ_i correspond to $K_i(\hat{\gamma})$ under the above K -theory isomorphism.

Proof. Let A be a unital UCT Kirchberg algebra in Cuntz standard form with $K_i(A) \cong G_i$ for $i = 0, 1$. As the G_i are uniquely p -divisible, we have $A \cong A \otimes M_{p^\infty}$. By 2.11, we can find a Rokhlin action $\alpha : \mathbb{Z}_p \curvearrowright A$ such that $K_i(\alpha) = \sigma_i$. By 4.1, we obtain $K_i(A \rtimes_\alpha \mathbb{Z}_p) = 0$. Moreover, since α has the Rokhlin property, $A \rtimes_\alpha \mathbb{Z}_p$ is a unital UCT Kirchberg algebra. This implies that $\mathcal{O}_2 \cong A \rtimes_\alpha \mathbb{Z}_p$. Under this identification, the dual action $\hat{\alpha}$ gives rise to a locally $\{A\}$ -representable (see 3.4) action $\gamma : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ with

$$\mathcal{O}_2^\gamma \cong (A \rtimes_\alpha \mathbb{Z}_p)^{\hat{\alpha}} \cong A$$

and

$$\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_p \cong (A \rtimes_\alpha \mathbb{Z}_p) \rtimes_{\hat{\alpha}} \mathbb{Z}_p \cong M_p \otimes A \cong A.$$

Moreover, $K_i(\hat{\gamma})$ corresponds to $K_i(\hat{\alpha}) = K_i(\alpha)$, which corresponds to σ_i . \square

Example 4.5. A special case of the above is $G_0 = \mathbb{Z}[\frac{1}{2}]$ with $\sigma_0 = -\text{id}$ and $G_1 = 0$. This yields a locally UCT Kirchberg-representable action γ on \mathcal{O}_2 with $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_2 \cong M_{2^\infty} \otimes \mathcal{O}_\infty^{\text{st}}$. This action actually coincides with the action constructed in [17, 4.7].

Remark. Example 4.4 can be generalized further, as one may delete the UCT assumption with the help of 2.12 to cover an (at least a priori) greater class of possible fixed point algebras. As this is not entirely obvious, we sketch how this can be done.

Remark 4.6. Let G be a finite group and let A be a separable, nuclear C^* -algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A is either unital, stable or has stable rank one. Let $\gamma : G \curvearrowright A$ be a Rokhlin action and let $\kappa : G \rightarrow KK(A, A)^{-1}$, $\kappa(g) = KK(\gamma_g)$ be the induced homomorphism to KK . For every separable C^* -algebra B , we have an isomorphism

$$KK(A \rtimes_\gamma G, B) \cong \{x \in KK(A, B) \mid \kappa(g) \otimes x = x \text{ for all } g \in G\}$$

of abelian groups.

Proof. We omit the most technical details. The key to proving this is 2.15. If one plugs in the Milnor sequence (see [4, 21.5.2]) for the functor $KK(_, B)$, one can describe the abelian group $KK(A \rtimes_\gamma G, B)$ as the stationary inverse limit

$$KK(A \rtimes_\gamma G, B) \cong \varprojlim \{KK(A, B), \psi\},$$

where $\psi(x) = \sum_{g \in G} \kappa(g) \otimes x$. For this argument, one also has to use unique $|G|$ -divisibility to see that the image of ψ^2 is equal to the image of ψ (since $\psi^2 = |G| \cdot \psi$), in order to deduce that $\varprojlim^1 \{KK(A, B), \psi\}$ vanishes for all B . Using unique $|G|$ -divisibility of the group $KK(A, B)$ one more time, one can see that this limit yields the group in the assertion. \square

Example 4.7. Let $p \geq 2$ be a natural number. Let A be a unital Kirchberg algebra in Cuntz standard form with $A \cong M_{p^\infty} \otimes A$. Assume that $\kappa \in KK(A, A)$ is an element satisfying the equations $1 = \kappa^p$ and $0 = \sum_{j=0}^{p-1} \kappa^j$. Then there exists a locally $\{A\}$ -representable action $\gamma : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ such that $\mathcal{O}_2^\gamma \cong \mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_p \cong A$ and such that $KK(\hat{\gamma}) = \kappa$.

Proof. We use 2.12 to find a Rokhlin action $\alpha : \mathbb{Z}_p \curvearrowright A$ with $KK(\alpha) = \kappa$. The previous Remark 4.6 allows one to compute the KK -groups for the crossed product $A \rtimes_\alpha \mathbb{Z}_p$. We have

$$KK(A \rtimes_\alpha \mathbb{Z}_p, B) \cong \{x \in KK(A, B) \mid \kappa \otimes x = x\}$$

as abelian groups for every separable C*-algebra B . If one keeps in mind the second equality for κ from above, one obtains

$$x = p \cdot \frac{x}{p} = \left(\sum_{j=0}^{p-1} \kappa^j \right) \cdot \frac{x}{p} = 0$$

for all $x \in KK(A \rtimes_\alpha \mathbb{Z}_p, A \rtimes_\alpha \mathbb{Z}_p)$. Hence we have $A \rtimes_\alpha \mathbb{Z}_p \cong \mathcal{O}_2$. The action γ that corresponds to $\hat{\alpha}$ under this identification, is locally $\{A\}$ -representable by 3.4. We have

$$\mathcal{O}_2^\gamma \cong (A \rtimes_\alpha \mathbb{Z}_p)^{\hat{\alpha}} \cong A$$

and

$$\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_p \cong (A \rtimes_\alpha \mathbb{Z}_p) \rtimes_{\hat{\alpha}} \mathbb{Z}_p \cong M_p \otimes A \cong A.$$

Moreover, we see that $KK(\hat{\gamma}) = KK(\hat{\alpha}) = KK(\alpha) = \kappa$. \square

Definition 4.8. Let \mathfrak{C}_R be the class of C*-algebras that are expressible as an inductive limit of 1-dimensional NCCW complexes with trivial K_1 -groups. Note that [38] provides a classification result for \mathfrak{C}_R . In particular, [38, 6.2.4] reduces the classifying invariant in the simple case to the K_0 -group, the tracial simplex and the pairing between these.

Remark 4.9 (compare to [31, 5.6]). A result of Elliott [9, 5.2.1 and 5.2.2] shows that there exists a simple, stably projectionless C*-algebra B in \mathfrak{C}_R with

$$(K_0(B), (T(B), T_1(B)), r_B) = (\mathbb{Z}, (\mathbb{R}^+, \emptyset), 0).$$

By [32, 5.2], there exists a hereditary subalgebra \mathcal{B}^{st} of B such that \mathcal{B}^{st} has a unique tracial state τ and no unbounded traces. Since \mathcal{B}^{st} is stably isomorphic to B by Brown's theorem, we have

$$(K_0(\mathcal{B}^{\text{st}}), (T(\mathcal{B}^{\text{st}}), T_1(\mathcal{B}^{\text{st}})), r_{\mathcal{B}^{\text{st}}}) = (\mathbb{Z}, (\mathbb{R}^+, \{\tau\}), 0).$$

By Robert's classification theorem [38], \mathcal{B}^{st} is (up to isomorphism) the unique simple, stably projectionless C^* -algebra in \mathfrak{C}_R with this data.

Remark 4.10. The notation \mathcal{B}^{st} is justified, as \mathcal{B}^{st} should be thought of a stably projectionless analogue of $\mathcal{O}_{\infty}^{\text{st}}$. A unital Kirchberg algebra A is in Cuntz standard form, i.e. $[\mathbf{1}_A]_0$ is trivial in $K_0(A)$, if and only if $A \cong A \otimes \mathcal{O}_{\infty}^{\text{st}}$. Similarly, one might be tempted to conjecture that a separable, simple, nuclear, stably projectionless, \mathcal{Z} -stable C^* -algebra A has a trivial pairing map $0 = r_A : T(A) \rightarrow S(K_0(A))$ if and only if $A \cong A \otimes \mathcal{B}^{\text{st}}$.

Remark 4.11. As we need it in the next example, we remark that a similar formula as in 4.1 holds for the non-unital (and non-simple) case as well. To be more precise, we have the following:

Let G be a finite group and A a separable C^* -algebra with $A \cong M_{|G|^\infty} \otimes A$. Let $\alpha : G \curvearrowright A$ be a Rokhlin action. Then

$$K_i(A^\alpha) \cong K_i(A \rtimes_\alpha G) \cong \bigcap_{g \in G} \ker(\text{id} - K_i(\alpha_g)) \quad \text{for } i = 0, 1.$$

As this follows easily from 2.15 and 2.16 after stabilizing with the trivial action on the compacts \mathcal{K} , we omit the proof.

The following example is essentially the same as the one treated in [31, 5.6].

Example 4.12. Let \mathcal{W} denote the so-called Razak-Jacelon algebra from [21]. There exists an action $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{W}$ with

$$\mathcal{W}^\gamma \cong \mathcal{W} \rtimes_\gamma \mathbb{Z}_2 \cong M_{2^\infty} \otimes \mathcal{B}^{\text{st}}$$

and such that γ is locally \mathcal{W}^γ -representable.

Proof. By Robert's classification theorem, there exists an automorphism $\beta \in \text{Aut}(M_{2^\infty} \otimes \mathcal{B}^{\text{st}})$ with $K_0(\beta) = -\text{id}$. As β^2 is trivial on the Elliott invariant, β^2 is approximately inner. Note that all C^* -algebras in \mathfrak{C}_R have stable rank one. Hence, we can apply 2.7 and get a Rokhlin action $\alpha : \mathbb{Z}_2 \curvearrowright M_{2^\infty} \otimes \mathcal{B}^{\text{st}}$ with $K_0(\alpha) = -\text{id}$. As \mathfrak{C}_R is closed under taking crossed products with Rokhlin actions of finite groups (see [41, Section 5]), $(M_{2^\infty} \otimes \mathcal{B}^{\text{st}}) \rtimes_\alpha \mathbb{Z}_2$ is in \mathfrak{C}_R and has trivial K -theory by 4.11. Moreover, it has a unique tracial state and no unbounded traces. Hence it is isomorphic to the Razak-Jacelon algebra \mathcal{W} . Under this identification, the dual action $\hat{\alpha}$ gives rise to a locally $\{M_{2^\infty} \otimes \mathcal{B}^{\text{st}}\}$ -representable (see 3.4) action $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{W}$ with

$$\mathcal{W}^\gamma \cong ((M_{2^\infty} \otimes \mathcal{B}^{\text{st}}) \rtimes_\alpha \mathbb{Z}_2)^{\hat{\alpha}} \cong M_{2^\infty} \otimes \mathcal{B}^{\text{st}}$$

and

$$\mathcal{W} \rtimes_{\gamma} \mathbb{Z}_2 \cong ((M_{2\infty} \otimes \mathcal{B}^{\text{st}}) \rtimes_{\alpha} \mathbb{Z}_2) \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \cong M_2 \otimes M_{2\infty} \otimes \mathcal{B}^{\text{st}} \cong M_{2\infty} \otimes \mathcal{B}^{\text{st}}.$$

□

At last, we would like to show that one can reduce the UCT problem for separable, nuclear C*-algebras to a question about certain locally representable actions on \mathcal{O}_2 .

Remark 4.13 (see [33, 10.2]). Let $p \geq 2$ be a natural number. Let us pick a primitive p -th root of unity $\xi_p = \exp(2\pi i/p) \in \mathbb{C}$. Then the ring generated by \mathbb{Z} and ξ_p , written $\mathbb{Z}[\xi_p]$, coincides with the ring of integers in the number field $\mathbb{Q}(\xi_p)$. The additive subgroup of this ring is well-known to be free abelian, with rank equal to $[\mathbb{Q}(\xi_p) : \mathbb{Q}]$, which coincides with the value of Euler's phi-function at p

$$\varphi(p) = |\{j \in \{1, \dots, p\} \mid \gcd(j, p) = 1\}|.$$

For example, if p happens to be prime, then $\varphi(p) = p - 1$. Hence, we have $\mathbb{Z}[\xi_p] \cong \mathbb{Z}^{\varphi(p)}$ as abelian groups.

The following contains 4.5 as a special case and is simultaneously contained in 4.4 as a special case.

Proposition 4.14. *Let $p \geq 2$ be a natural number. Then there exists a locally UCT Kirchberg-representable action $\gamma_p : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ such that $A_p = \mathcal{O}_2 \rtimes_{\gamma_p} \mathbb{Z}_p$ is KK -equivalent to $M_{p\infty}^{\oplus \varphi(p)}$.*

Proof. Choose a unital UCT Kirchberg algebra A_p with K -theory

$$(K_0(A_p), [\mathbf{1}_{A_p}]_0, K_1(A_p)) \cong (\mathbb{Z}[\frac{1}{p}]^{\oplus \varphi(p)}, 0, 0).$$

By the UCT, A_p is in fact KK -equivalent to $M_{p\infty}^{\oplus \varphi(p)}$, since they have identical K -theory. Because of 4.13, $K_0(A_p)$ is isomorphic to the additive group of the ring $\mathbb{Z}[\frac{1}{p}, \xi_p]$. Under this identification, we obtain an order p automorphism $\sigma : K_0(A_p) \rightarrow K_0(A_p)$ by $x \mapsto \xi_p \cdot x$. Note that obviously $\ker(\text{id} - \sigma) = 0$.

Since the K -theory of A_p is uniquely p -divisible, we have $A_p \cong M_{p\infty} \otimes A_p$. By 2.11, there exists a Rokhlin action $\alpha : \mathbb{Z}_p \curvearrowright A_p$ with $K_0(\alpha) = \sigma$. Note that from the above observations, combined with 4.1 and Kirchberg-Phillips classification, it follows that $A_p \rtimes_{\alpha} \mathbb{Z}_p \cong \mathcal{O}_2$. Under this identification, the dual action $\gamma_p = \hat{\alpha} : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ yields a locally $\{A_p\}$ -representable (see 3.4) action with $\mathcal{O}_2 \rtimes_{\gamma_p} \mathbb{Z}_p \cong A_p$. This finishes the proof. □

The following is well-known:

Proposition 4.15. *Let $n \in \mathbb{N}$ be a natural number and A_1, \dots, A_n separable C*-algebras. Then each A_i satisfies the UCT if and only if $A_1 \oplus \dots \oplus A_n$ satisfies the UCT.*

Proposition 4.16. *Let A be a separable C^* -algebra, and let $p, q \geq 2$ be two relatively prime natural numbers. If both $M_{p^\infty} \otimes A$ and $M_{q^\infty} \otimes A$ satisfy the UCT, then so does A .*

Proof. One has that the C^* -algebra $Z_{p^\infty, q^\infty} \otimes A$ arises as an extension of $M_{p^\infty} \otimes A \oplus M_{q^\infty} \otimes A$ by $\mathcal{C}_0(0, 1) \otimes M_{p^\infty} \otimes M_{q^\infty} \otimes A$, which both satisfy the UCT by assumption. Hence also $Z_{p^\infty, q^\infty} \otimes A$ satisfies the UCT. Since the Jiang-Su algebra arises as a stationary inductive limit (with injective connecting maps) of Z_{p^∞, q^∞} , we have that also $\mathcal{Z} \otimes A \sim_{KK} A$ satisfies the UCT. \square

Theorem 4.17. *Let $p, q \geq 2$ be two prime numbers. The following are equivalent:*

- (1) *Every separable, nuclear C^* -algebra satisfies the UCT.*
- (2) *Every unital Kirchberg algebra satisfies the UCT.*
- (3) *If $\beta : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ and $\gamma : \mathbb{Z}_q \curvearrowright \mathcal{O}_2$ are pointwise outer, locally Kirchberg-representable actions, then both $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_p$ and $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_q$ satisfy the UCT.*
- (4) *If $\gamma : \mathbb{Z}_{pq} \curvearrowright \mathcal{O}_2$ is a pointwise outer, locally Kirchberg-representable action, then $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_{pq}$ satisfies the UCT.*

Proof. First we observe that the implications (1) \implies (2), (3), (4) are trivial. We will first show the implications (3) \implies (2) and (4) \implies (2):

(3) \implies (2): Assume that (2) is false. Then we can pick a unital Kirchberg algebra A that does not satisfy the UCT. By 4.15 and 4.16, it follows that either $A \otimes M_{p^\infty}^{\oplus(p-1)}$ or $A \otimes M_{q^\infty}^{\oplus(q-1)}$ does not satisfy the UCT.

Then it follows from 4.14 actions that either

$$(A \otimes \mathcal{O}_2) \rtimes_{\text{id}_A \otimes \gamma_p} \mathbb{Z}_p \cong A \otimes A_p \sim_{KK} A \otimes M_{p^\infty}^{\oplus(p-1)}$$

or

$$(A \otimes \mathcal{O}_2) \rtimes_{\text{id}_A \otimes \gamma_q} \mathbb{Z}_q \cong A \otimes A_q \sim_{KK} A \otimes M_{q^\infty}^{\oplus(q-1)}$$

does not satisfy the UCT. Since both γ_p and γ_q are pointwise outer and locally Kirchberg-representable and $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, this gives a counterexample to (3).

(4) \implies (2): Assume that (2) is false. Then we can pick a unital Kirchberg algebra A that does not satisfy the UCT. By 4.15 and 4.16, it follows that either $(A \otimes M_{p^\infty})^{\oplus(p-1) \cdot q}$ or $(A \otimes M_{q^\infty})^{\oplus(q-1) \cdot p}$ does not satisfy the UCT. Now by [12], there exist two pointwise outer, locally $\{\mathcal{O}_\infty\}$ -representable actions $\alpha_p : \mathbb{Z}_p \curvearrowright \mathcal{O}_\infty$ and $\alpha_q : \mathbb{Z}_q \curvearrowright \mathcal{O}_\infty$ with

$$\mathcal{O}_\infty \rtimes_{\alpha_p} \mathbb{Z}_p \sim_{KK} C^*(\mathbb{Z}_p) \cong \mathbb{C}^p \quad \text{and} \quad \mathcal{O}_\infty \rtimes_{\alpha_q} \mathbb{Z}_q \sim_{KK} C^*(\mathbb{Z}_q) \cong \mathbb{C}^q.$$

Recalling that $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, we obtain two pointwise outer, locally Kirchberg-representable actions $\gamma_p \otimes \alpha_q : \mathbb{Z}_{pq} \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$ and $\gamma_q \otimes \alpha_p : \mathbb{Z}_{pq} \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$ with

$$\mathcal{O}_2 \rtimes_{\gamma_p \otimes \alpha_q} \mathbb{Z}_{pq} \cong A_p \otimes (\mathcal{O}_\infty \rtimes_{\alpha_q} \mathbb{Z}_q) \sim_{KK} M_{p^\infty}^{\oplus(p-1) \cdot q}$$

and

$$\mathcal{O}_2 \rtimes_{\gamma_q \otimes \alpha_p} \mathbb{Z}_{pq} \cong A_q \otimes (\mathcal{O}_\infty \rtimes_{\alpha_p} \mathbb{Z}_p) \sim_{KK} M_{q^\infty}^{\oplus(q-1)p}.$$

In particular, it follows that either

$$(A \otimes \mathcal{O}_2) \rtimes_{\text{id}_A \otimes \gamma_p \otimes \alpha_q} \mathbb{Z}_{pq} \sim_{KK} (A \otimes M_{p^\infty})^{\oplus(p-1)q}$$

or

$$(A \otimes \mathcal{O}_2) \rtimes_{\gamma_q \otimes \alpha_p} \mathbb{Z}_{pq} \sim_{KK} (A \otimes M_{q^\infty})^{\oplus(q-1)p}$$

does not satisfy the UCT. Hence this gives a counterexample to (4).

Finally, let us show the implication (2) \implies (1). We stress that this is already well-known and due to Kirchberg (see [24, 2.17] for an even stronger assertion), but we repeat the argument for the reader's convenience:

Let A be a separable, nuclear C*-algebra. By adding a unit, if necessary, we can assume that A is unital. (This procedure does not change if A is in the UCT class or not.) We show that A is KK -equivalent to a unital Kirchberg algebra, which immediately confirms the implication (2) \implies (1).

For this, we moreover assume that $A \cong A \otimes \mathcal{O}_\infty^{\text{st}}$. If this is not the case, we can stabilize with $\mathcal{O}_\infty^{\text{st}}$, since this does not change the KK -equivalence class. If $A \cong A \otimes \mathcal{O}_\infty^{\text{st}}$, there exists a unital embedding $\iota : \mathcal{O}_2 \rightarrow A$. Let us explicitly choose two isometries $s_1, s_2 \in A$ with $\mathbf{1} = s_1 s_1^* + s_2 s_2^*$. By Kirchberg's embedding theorem (see [25]), there exists a unital embedding $\kappa : A \rightarrow \mathcal{O}_2$. Now define an endomorphism

$$\varphi : A \rightarrow A \quad \text{via} \quad \varphi(x) = s_1 x s_1^* + s_2 (\iota \circ \kappa)(x) s_2^*.$$

Consider the inductive limit $B = \varinjlim \{A, \varphi\}$. Then B is clearly again separable, unital, nuclear and \mathcal{O}_∞ -absorbing. Since for all $x \neq 0$, the element

$$s_2^* \varphi(x) s_2 = \iota \circ \kappa(x)$$

is the image of the full element $\kappa(x) \in \mathcal{O}_2$, it follows that $\varphi(x)$ is also full. Hence B is simple.

Lastly, it is immediate that $KK(\varphi) = 1 + KK(\iota \circ \kappa) = 1$, since $\iota \circ \kappa$ factors through \mathcal{O}_2 . In particular, the connecting maps of this inductive system induce KK -equivalences. Hence it follows that the canonical embedding $\varphi_\infty : A \rightarrow B$ induces a KK -equivalence. This is implied by [8, 2.4], which basically boils down to plugging in the Milnor sequence [4, 21.5.2] for the functor $KK(_, B)$ in this situation.

To summarize, we have found a unital Kirchberg algebra B that is KK -equivalent to A . This finishes the proof. \square

Remark 4.18. From the proof of 4.17, we can say a bit more in the case that we consider actions of only one natural number. Namely, we get the following:

Let $p \geq 2$ be a natural number. Assume that for all pointwise outer, locally Kirchberg-representable actions $\gamma : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$, the crossed product $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_p$ satisfies the UCT. Then all separable, nuclear and M_{p^∞} -absorbing C*-algebras satisfy the UCT.

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